Informal Introduction to Spectral Risk Measures

Stephen J. Mildenhall

May 2018

ST. JOHN'S UNIVERSITY
Tobin College of Business
School of Risk Management

Don said “Risk Measure = Expression of Risk Preference”

…but what is a Risk Preference?

Risk Preferences

Rational actors

- Prefer more to less

Risk Preferences

Rational actors

- Prefer more to less
- Prefer certainty to uncertainty

Prefer More To Less

Sounds simple, but

Prefer More To Less

Sounds simple, but

- Diminishing marginal utility
- Preference **relative** to a wealth level
- Not well suited to **corporations**

Prefer Certainty to Uncertainty

Risk multifaceted

- Process
- Parameter
- Uncertainty
- Ambiguity
- Pure
- Speculative

Prefer Certainty to Uncertainty

Risk multifaceted

- Process
- Parameter
- Uncertainty
- Ambiguity
- Pure
- Speculative

- Modeling assets: large positive good
- Modeling losses: large positive bad

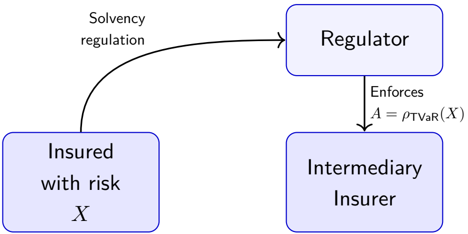
Risk Measure ρ Quantifies Risk Preferences

$$\text{Prefer } X \text{ to } Y \iff \rho(X) \le \rho(Y)$$

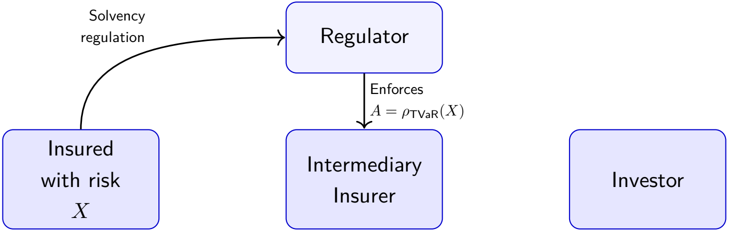
- Simple
- Consistent
- Applies to pricing
- Applies to risk capital

Risk Measure**s** Determine Assets, Prices, and Capital

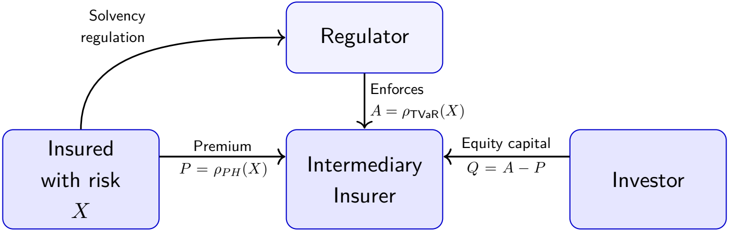
Risk Measure**s** Determine Assets, Prices, and Capital



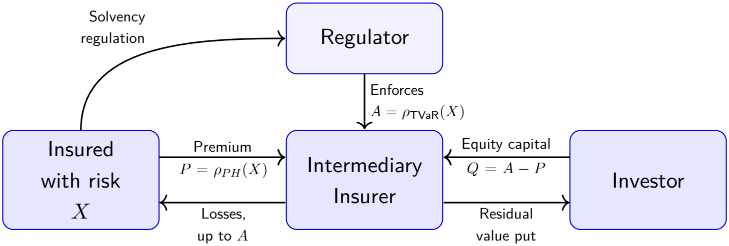
Risk Measure**s** Determine Assets, Prices, and Capital



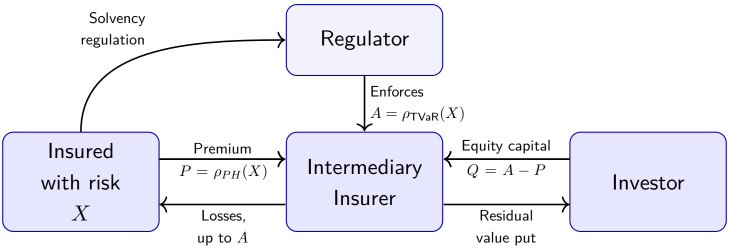
Risk Measure**s** Determine Assets, Prices, and Capital



Risk Measures Determine Assets, Prices, and Capital



Risk Measures Determine Assets, Prices, and Capital



- Regulator risk measure determines and enforces adequate risk bearing capacity A, e.g. with TVaR
- Market risk measure determines split of A into premium and equity

The Thin Layer Trick

Distortion Functions Price Thin Layers

Simplifying idea

- Break pricing problem into sub-problems of pricing thin layers
- Add!

Distortion Functions Price Thin Layers

Simplifying idea

- Break pricing problem into sub-problems of pricing thin layers
- Add!

Why simplifying?

- Thin layers only have total losses, no partial losses
- **Risk** of thin layer completely described by **one number**, called
  - Exceedance probability (EP) S, or
  - Probability of attachment, or
  - Expected loss (EL)

Distortion Functions Price Thin Layers

Linking risk and price

- **Price** of thin layer also described by **one number**, called
  - Rate-on-line (ROL), or
  - Risk adjusted or distorted probability, or
  - State-price

Distortion Functions Price Thin Layers

Linking risk and price

- **Price** of thin layer also described by **one number**, called
  - Rate-on-line (ROL), or
  - Risk adjusted or distorted probability, or
  - State-price
- **Distortion function** g : thin layer risk $\mapsto$ price captures relationship between risk and price
  - g is a function $[0,1] \to [0,1]$
  - Risk averse implies $g(s) \geq s$ for all $s \in [0,1]$

Distortion Function to Risk Measure

- Associate a **risk measure** ρ_g to a distortion function g by analogy with $\mathsf{E}(X)$

$$\begin{aligned}\mathsf{E}(X) &= \int_0^\infty S(x)dx \\ &= \int_0^\infty xf(x)dx \\ &= \int_0^1 F^{-1}(p)dp\end{aligned}$$

Distortion Function to Risk Measure

- Associate a **risk measure** ρ_g to a distortion function g by analogy with $\mathsf{E}(X)$

$$\begin{aligned}\mathsf{E}(X) &= \int_0^\infty S(x)dx \\ &= \int_0^\infty xf(x)dx \\ &= \int_0^1 F^{-1}(p)dp\end{aligned}$$

$$\begin{aligned}\rho_g(X) &= \int_0^\infty g(S(x))dx \\ &= \int_0^\infty xg'(S(x))f(x)dx \\ &= \int_0^1 \mathsf{VaR}_p(X)g'(1-p)dp\end{aligned}$$

Distortion Function to Risk Measure

- Associate a **risk measure** ρ_g to a distortion function g by analogy with $\mathsf{E}(X)$

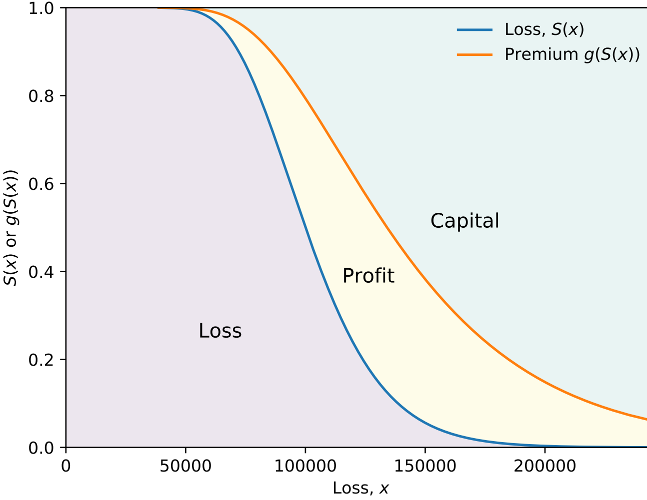
$$\mathsf{E}(X) = \int_0^\infty S(x)dx \qquad \rho_g(X) = \int_0^\infty g(S(x))dx$$

$$= \int_0^\infty xf(x)dx \qquad = \int_0^\infty xg'(S(x))f(x)dx$$

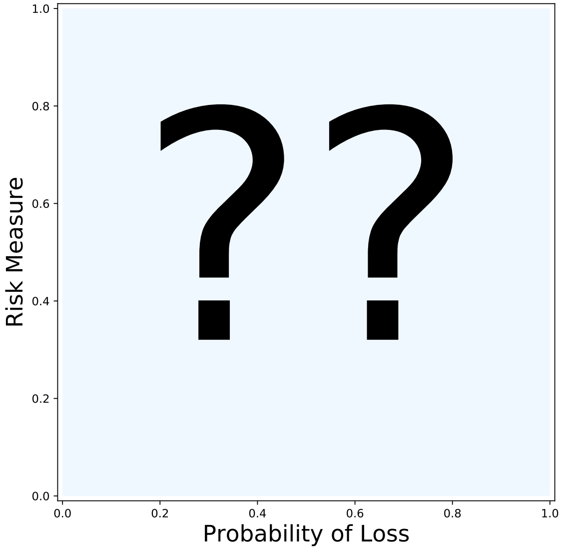
$$= \int_0^1 F^{-1}(p)dp \qquad = \int_0^1 \mathsf{VaR}_p(X)g'(1-p)dp$$

- Function g' on lower right measures care/care-more along the risk spectrum p, hence **spectral risk measure**
- $\mathsf{E}(X)$ corresponds to ρ_g with $g(s) = s$ the identity

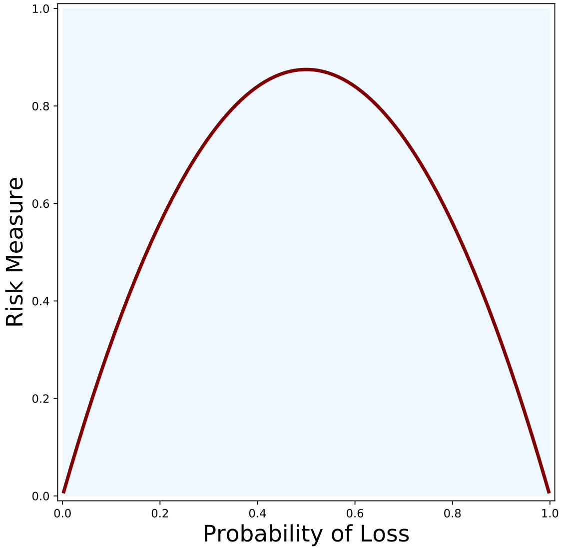
Total Risk: Summing Over Thin Layers Using $g(S(x))$



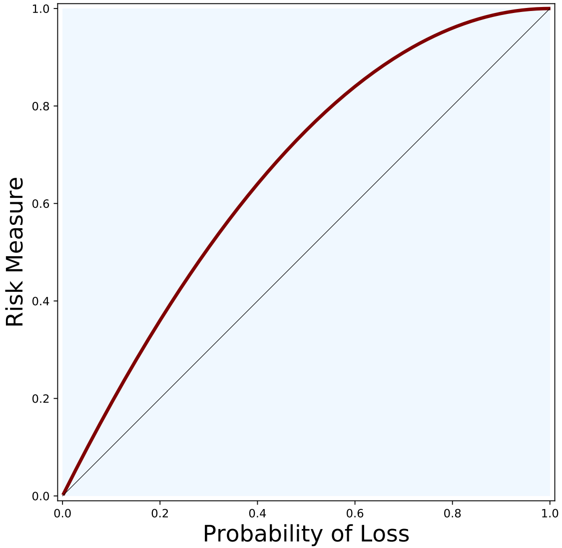
Relationship Between Risk and Price



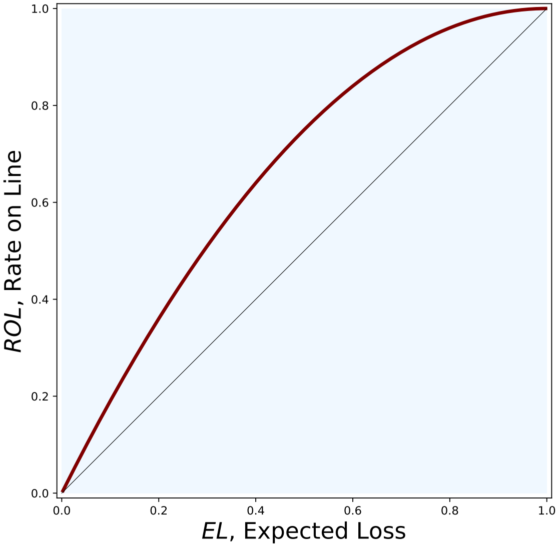
Risk: Entropy or Standard Deviation?



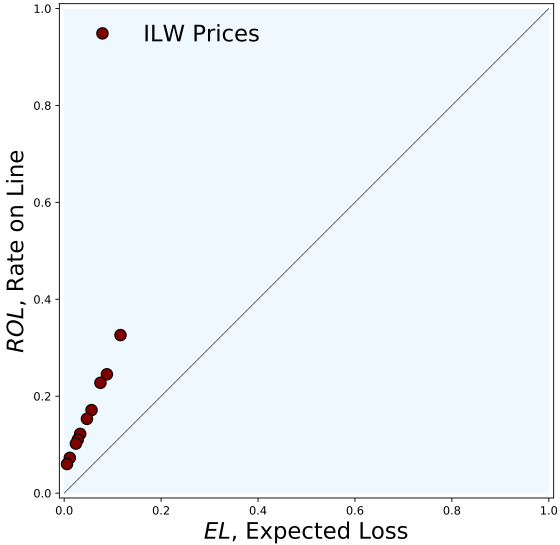
Risk **Measure**: Encompasses **Volume** and **Volatility**



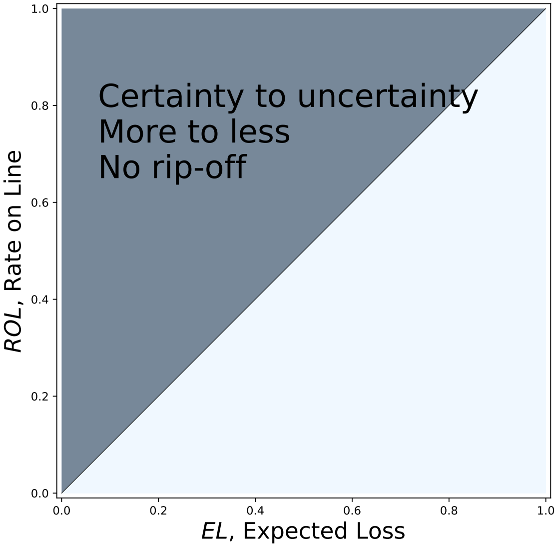
Thin Layer: Probability of Loss=EL and Risk=Price=ROL



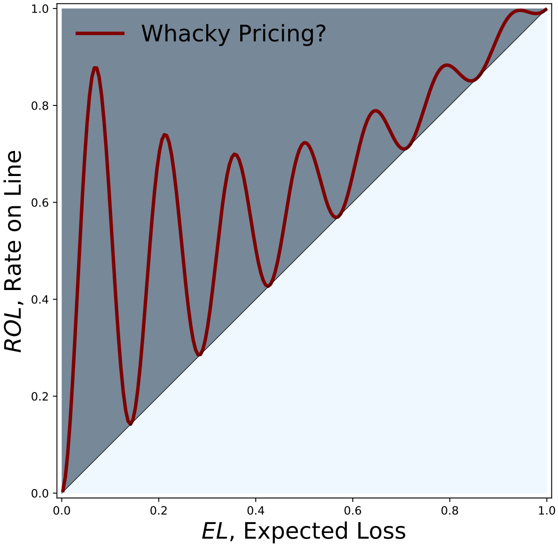
Empirical Data Guides Choice of Pricing Function



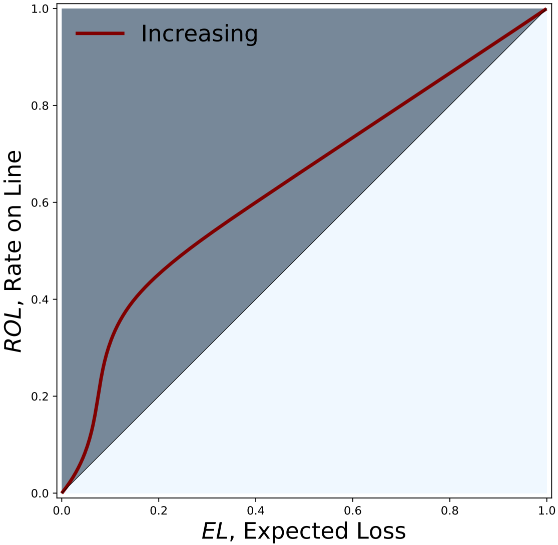
Restrictions on Possible Distortion / Pricing Functions



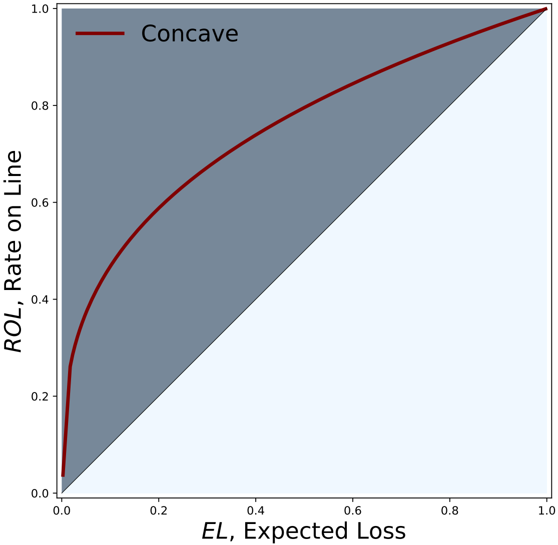
Restrictions on Pricing Functions: $0 \mapsto 0$, $1 \mapsto 1$



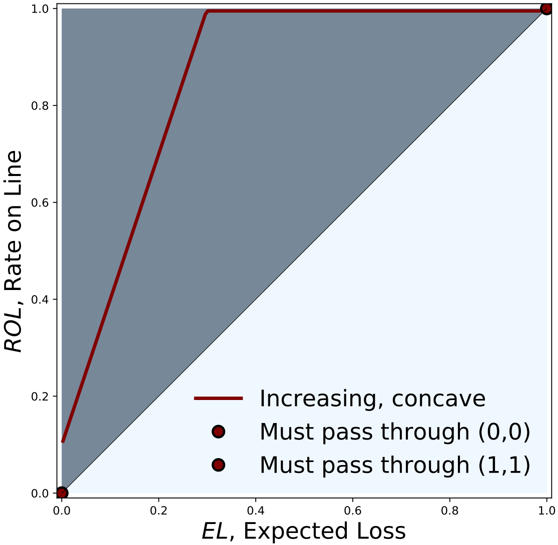
Restriction: Increasing $\iff$ Monotone



Restriction: Concave $\iff$ Subadditive



Four Restrictions Leave Great Flexibility



And With Great Flexibility, Comes Great Responsibility

John, over to you. . .